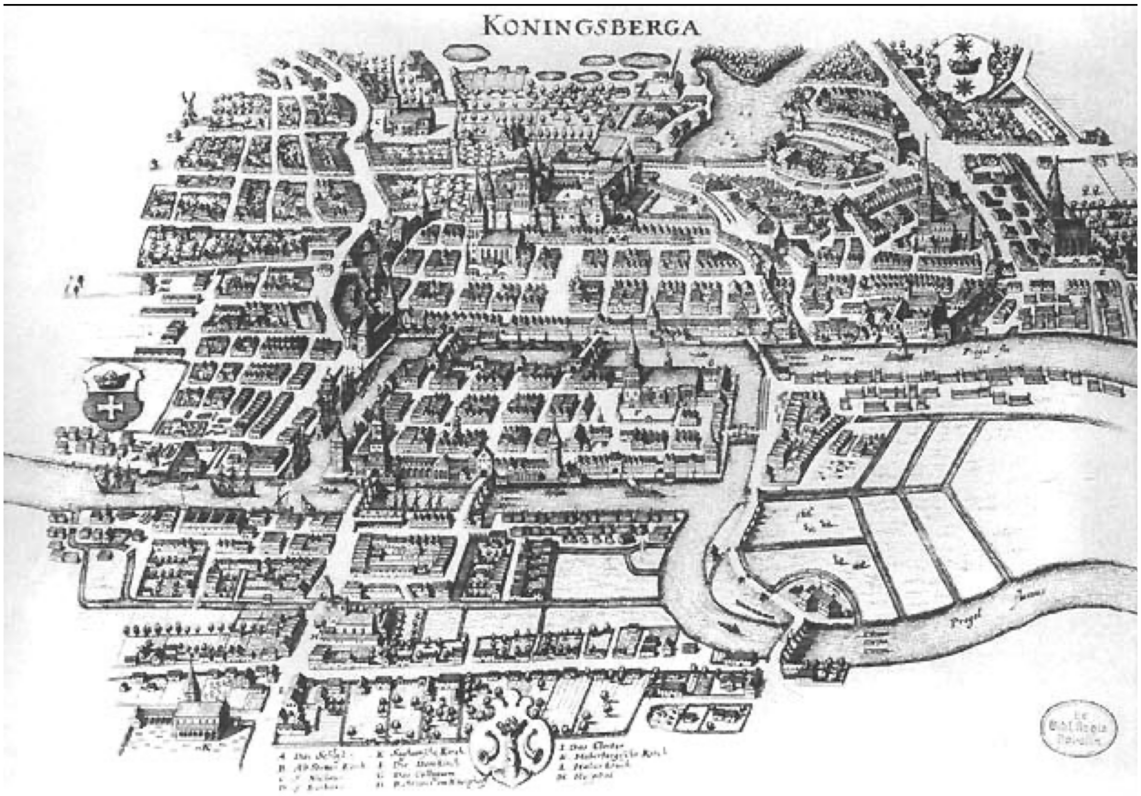
DAS PROBLEM DER KÖNIGSBERGER BRÜCKEN

HANSPETER KRAFT

Problem. *Gibt es einen Spaziergang durch Königsberg, bei dem jede der sieben Brücken genau einmal überschritten wird?*



Euler lernte das Problem durch den Danziger Bürgermeister Carl Leonhard Gottlieb Ehler kennen – wohl bei einer persönlichen Begegnung in Petersburg, wo Ehler 1734/35 auf diplomatischer Mission war, und korrespondierte darüber mit Ehler und Marinoni. Er behandelt es in der Arbeit [Eul36] mit dem Titel *«Solutio problematis ad geometriam situs pertinentis»*, welche 1736 in den Commentarii Academiae Scientiarum Imperialis Petropolitanae 8, p. 128–140 erschienen ist (→ pdf im Anhang). Interessanterweise hat sich Euler später nicht mehr mit dieser Problematik beschäftigt.

Zum Problem selbst haben B. Hopkins und R. Wilson einen schönen und lesenswerten Artikel geschrieben [HW04], mit vielen Details zur Geschichte des Problems und zu Eulers Briefwechsel in diesem Zusammenhang (→ pdf im Anhang).

In den Opera Omnia ist die Arbeit in Vol. 7 der Series I auf den Seiten 1–10 abgedruckt. In diesem Band, welcher von LOUIS GUSTAVE DU PASQUIER im Jahre 1923 herausgegeben wurde, gibt es ein längeres Vorwort, welches folgenden Abschnitt über diese Arbeit enthält.

II. LES RECREATIONS MATHEMATIQUES

§ 2. Ce sont presque toujours des occasions spéciales qui ont amené EULER à s'occuper de ces sujets. Le mémoire 53, *Solutio problematis ad geometriam situs pertinentis*, qui ouvre le volume, fut présenté le 26 août 1735 à l'Académie des sciences de Saint-Pétersbourg, mais publié seulement en 1741. EULER y résout le fameux problème des ponts de Königsberg. Un piéton peut-il diriger sa promenade de manière à traverser chacun des sept ponts, mais chacun seulement une fois? EULER en prouve l'impossibilité, puis termine le mémoire par la démonstration de ce curieux théorème valable quels que soient les ramifications du fleuve, le nombre des îlots, le nombre et la position des ponts: s'il y a plus de deux régions dans lesquelles on peut pénétrer par un nombre impair de ponts, un chemin répondant aux conditions du problème n'existe pas. S'il y a au plus deux régions dans lesquelles on peut pénétrer par un nombre impair de ponts, un tel chemin existe, à condition que le point de départ soit pris dans l'une de ces régions-là. Enfin, il y a toujours une solution, où que se trouve le point de départ, lorsque la distribution des ponts et des îles est telle que c'est toujours par un nombre pair de ponts que chaque région est reliée à ses voisines. Il y aurait donc toujours une solution, si les conditions du problème exigeaient que chaque pont fût traversé deux fois, et deux fois seulement.

Une traduction française de ce mémoire EULÉRIEN, due à E. COUPY, a paru dans les Nouvelles Annales de mathématiques 10, 1851, p. 106—119, sous le titre *Solution d'un problème appartenant à la géométrie de situation*, et une autre traduction française se trouve en manuscrit dans la bibliothèque de l'observatoire astronomique d'Uccle près Bruxelles. On trouvera une analyse du mémoire EULÉRIEN dans les Nova Acta eruditorum, Lipsiae 1751, p. 593—594. Il fut réimprimé dans les Commentarii academiae scientiarum Petropolitanae 8, editio nova, Bononiae 1752, p. 116—126 plus une table.

Aus der Arbeit selbst erfährt man, dass EULER das Problem als Bestandteil einer «noch heute weitgehend unbekannten Disziplin» betrachtete, die LEIBNIZ als «Geometrie der Lage» bezeichnet hatte – einer Geometrie ohne quantitative Betrachtungen, also mehr oder weniger das, was man heute als Topologie bezeichnet.

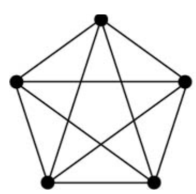
Die Lösung des Problems, also die notwendige und hinreichende Bedingung für die Existenz eines solchen Spazierganges, steht in §20 der Arbeit:

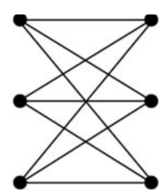
Si fuerint plures duabus regiones, ad quas ducentium pontium numerus est impar, tum certo affirmari potest talem transitum non dari.

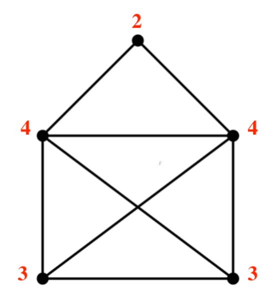
Si autem ad duas tantum regiones ducentium pontium numerus est impar, tunc transitus fieri poterit, si modo cursus in altera harum regionum incipiatur.

Si denique nulla omnino fuerit regio, ad quam pontes numero impares conducant, tum transitus desiderato modo institui poterit, in quacunque regione ambulandi initium ponatur.

In der modernen Sprache der *Graphentheorie* kann man das Problem folgendermassen umformulieren. Ein *Graph* Γ besteht aus einer endlichen Anzahl von Punkten (= Knoten), die mit Strecken (= Kanten) verbunden sind.







Jedem Knoten des Graphen Γ wird eine Zahl zugeordnet, die *Valenz des Knotens*, nämlich die Anzahl der Kanten, welche von diesem Knoten ausgehen. Im linken Bild sind alle Valenzen 4, in der Mitte alle 3, und beim *Nikolaushaus* rechts kommen die Valenzen 2, 3 und 4 vor.[1]

Einen Weg durch den Graphen Γ nennen wir einen *Euler-Weg*, wenn jede Kante genau einmal durchlaufen wird. Ein Weg heisst *geschlossen*, wenn der Anfangspunkt mit dem Endpunkt übereinstimmt.

Nun können wir das Resultat von EULER folgendermassen formulieren.

Satz. *In einem Graphen gibt es genau dann einen geschlossenen Euler-Weg, wenn die Valenz in jedem Knoten gerade ist. Es gibt genau dann einen nicht geschlossenen Euler-Weg, wenn die Valenz in allen Knoten gerade ist bis auf zwei Ausnahmen, welche Anfangs- und Endpunkt des Weges sind.*

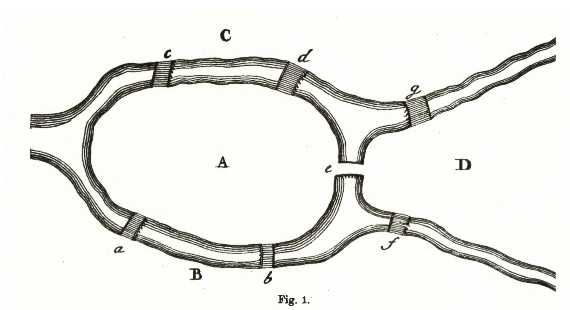
[1]Die Begriffe «Graph» und «Valenz» kommen bei EULER noch nicht vor.

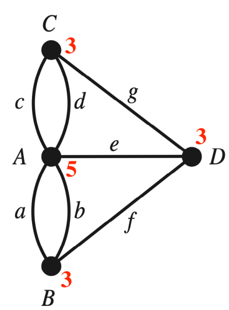
Zum Beweis. (a) Gibt es einen Euler-Weg, so muss von jedem Knoten eine gerade Anzahl von Kanten ausgehen, mit Ausnahme von Anfangs- und Endpunkt, falls diese nicht zusammenfallen, denn in einem Durchgangsknoten gibt es zu jeder ankommenden Kante eine abgehende Kante. Im Anfangspunkt ist diese Anzahl aber ungerade, falls er vom Endpunkt verschieden ist, und ebenso im Endpunkt.

(b) Der Nachweis, dass es bei gerader Valenz an allen Knoten einen geschlossenen Euler-Weg gibt, ist etwas schwieriger, denn es kann verschiedene solche Wege geben. Euler gibt dafür in § 21 ein recht kurzes Argument, wobei er dem Leser einiges überlässt, siehe [HW04, Paragraph 21, Seite 206]. Wir empfehlen dem Leser, einen solchen Beweis selber zu erarbeiten. □

Wenden wir den Satz auf die obigen drei Beispiele an, so folgt, dass es beim linken Beispiel einen Euler-Weg gibt, beim Beispiel in der Mitte kein Euler-Weg existiert, und beim Beispiel rechts gibt es einen Euler-Weg mit Anfangs- und Endpunkt bei den Knoten mit Valenz 3.

Der Königsberger Brückengraph hat nun folgende folgende Gestalt. Dabei sind die Stadtteile die Knoten und die Brücken entsprechen den Kanten.





Die Valenzen (im rechten Bild rot eingetragen) sind alle ungerade, also gibt es keinen Euler-Weg, und damit auch keinen Spaziergang durch Königsberg, welcher jede Brücke genau einmal überquert!

Literatur

[Eul36] Leonhard Euler, *Solutio problematis ad geometriam situs pertinentis*, Commentarii Academiae Scientiarum Imperialis Petropolitanae **8** (1736), 128–140, E 15. Opera Omnia Ser. I, Vol. 7. p. 1–10.

[HW04] Brian Hopkins and Robin J. Wilson, *The Truth about Königsberg*, The College Mathematics Journal **35** (2004), no. 3, 198–207.

The Truth about Königsberg

Brian Hopkins and Robin J. Wilson

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Euler's 1736 paper on the bridges of Königsberg is widely regarded as the earliest contribution to graph theory—yet Euler's solution made no mention of graphs. In this paper we place Euler's views on the Königsberg bridges problem in their historical context, present his method of solution, and trace the development of the present-day solution.

What Euler didn't do

A well-known recreational puzzle concerns the bridges of Königsberg. It is claimed that in the early eighteenth century the citizens of Königsberg used to spend their Sunday afternoons walking around their beautiful city. The city itself consisted of four land areas separated by branches of the river Pregel over which there were seven bridges, as illustrated in Figure 1. The problem that the citizens set themselves was to

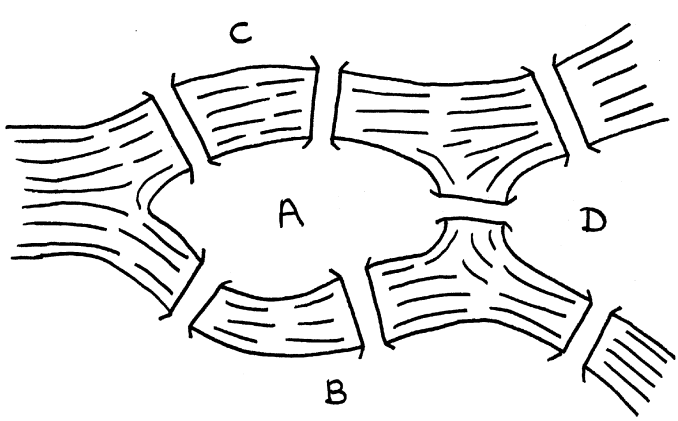

Figure 1. Königsberg

walk around the city, crossing each of the seven bridges exactly once and, if possible, returning to their starting point.

If you look in some books on recreational mathematics, or listen to some graph-theorists who should know better, you will 'learn' that Leonhard Euler investigated the Königsberg bridges problem by drawing a graph of the city, as in Figure 2, with a vertex representing each of the four land areas and an edge representing each of the seven bridges. The problem is then to find a trail in this graph that passes along each edge just once.

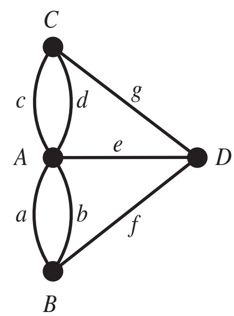

Figure 2. The Königsberg graph

But Euler didn't draw the graph in Figure 2—graphs of this kind didn't make their first appearance until the second half of the nineteenth century. So what exactly did Euler do?

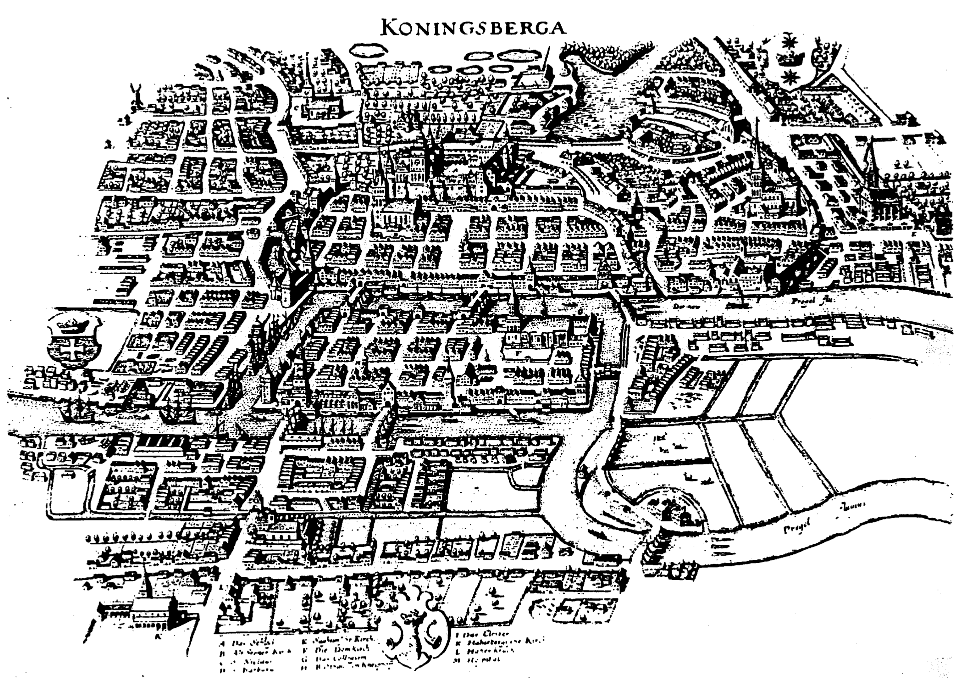

Figure 3. Seventeenth-century Königsberg

The Königsberg bridges problem

In 1254 the Teutonic knights founded the Prussian city of Königsberg (literally, king's mountain). With its strategic position on the river Pregel, it became a trading center and an important medieval city. The river flowed around the island of Kneiphof (literally, pub yard) and divided the city into four regions connected by seven bridges: Blacksmith's bridge, Connecting bridge, High bridge, Green bridge, Honey bridge, Merchant's bridge, and Wooden bridge: Figure 3 shows a seventeenth-century map of the city. Königsberg later became the capital of East Prussia and more recently became the Russian city of Kaliningrad, while the river Pregel was renamed Pregolya.

In 1727 Leonhard Euler began working at the Academy of Sciences in St Petersburg. He presented a paper to his colleagues on 26 August 1735 on the solution of 'a problem relating to the geometry of position': this was the Königsberg bridges problem. He also addressed the generalized problem: given any division of a river into branches and any arrangement of bridges, is there a general method for determining whether such a route exists?

In 1736 Euler wrote up his solution in his celebrated paper in the *Commentarii Academiae Scientiarum Imperialis Petropolitanae* under the title 'Solutio problematis ad geometriam situs pertinentis' [**2**]; Euler's diagram of the Königsberg bridges appears in Figure 4. Although dated 1736, Euler's paper was not actually published until 1741, and was later reprinted in the new edition of the *Commentarii* (*Novi Acta Commentarii* ...) which appeared in 1752.

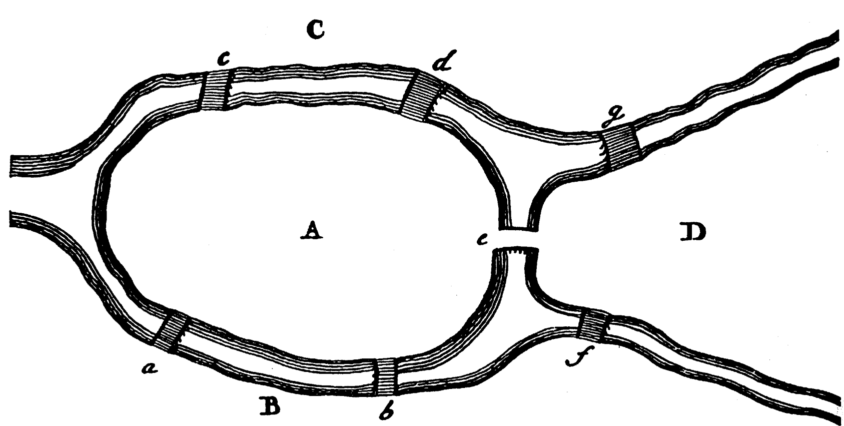

Figure 4. Diagram from Euler's 1736 paper

A full English translation of this paper appears in several places—for example, in [**1**] and [**6**]. The paper begins:

> 1. In addition to that branch of geometry which is concerned with distances, and which has always received the greatest attention, there is another branch, hitherto almost unknown, which Leibniz first mentioned, calling it the geometry of position [*Geometriam situs*]. This branch is concerned only with the determination of position and its properties; it does not involve distances, nor calculations made with them. It has not yet been satisfactorily determined what kinds of problem are relevant to this geometry of position, or what methods should be used in solving them. Hence, when a problem was recently mentioned which seemed geometrical but was so constructed that it did not require the measurement of distances, nor did calculation help at all, I had no doubt that it

> was concerned with the geometry of position—especially as its solution involved only position, and no calculation was of any use. I have therefore decided to give here the method which I have found for solving this problem, as an example of the geometry of position.
>
> 2. The problem, which I am told is widely known, is as follows: in Königsberg ...

This reference to Leibniz and the geometry of position dates back to 8 September 1679, when the mathematician and philosopher Gottfried Wilhelm Leibniz wrote to Christiaan Huygens as follows [**5**]:

> I am not content with algebra, in that it yields neither the shortest proofs nor the most beautiful constructions of geometry. Consequently, in view of this, I consider that we need yet another kind of analysis, geometric or linear, which deals directly with position, as algebra deals with magnitudes ...

Leibniz introduced the term *analysis situs* (or *geometria situs*), meaning the analysis of situation or position, to introduce this new area of study. Although it is sometimes claimed that Leibniz had vector analysis in mind when he coined this phrase (see, for example, [**8**] and [**11**]), it was widely interpreted by his eighteenth-century followers as referring to topics that we now consider 'topological'—that is, geometrical in nature, but with no reference to metrical ideas such as distance, length or angle.

Euler's Königsberg letters

It is not known how Euler became aware of the Königsberg bridges problem. However, as we shall see, three letters from the Archive Collection of the Academy of Sciences in St Petersburg [**3**] shed some light on his interest in the problem (see also [**10**]).

Carl Leonhard Gottlieb Ehler was the mayor of Danzig in Prussia (now Gdansk in Poland), some 80 miles west of Königsberg. He corresponded with Euler from 1735 to 1742, acting as intermediary for Heinrich Kühn, a local mathematics professor. Their initial communication has not been recovered, but a letter of 9 March 1736 indicates they had discussed the problem and its relation to the 'calculus of position':

> You would render to me and our friend Kühn a most valuable service, putting us greatly in your debt, most learned Sir, if you would send us the solution, which you know well, to the problem of the seven Königsberg bridges, together with a proof. It would prove to be an outstanding example of the calculus of position [*Calculi Situs*], worthy of your great genius. I have added a sketch of the said bridges ...

Euler replied to Ehler on 3 April 1736, outlining more clearly his own attitude to the problem and its solution:

> ... Thus you see, most noble Sir, how this type of solution bears little relationship to mathematics, and I do not understand why you expect a mathematician to produce it, rather than anyone else, for the solution is based on reason alone, and its discovery does not depend on any mathematical principle. Because of this, I do not know why even questions which bear so little relationship to mathematics are solved more quickly by mathematicians than by others. In the meantime, most noble Sir, you have assigned this question to the geometry of position, but I am ignorant as to what this new discipline involves, and as to which types of problem Leibniz and Wolff expected to see expressed in this way ...

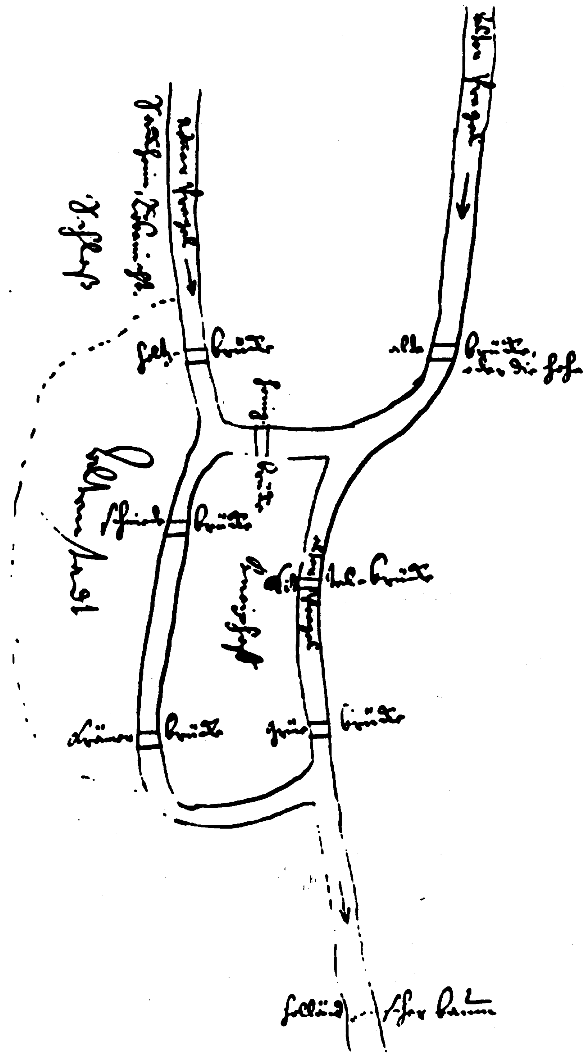

Figure 5. Ehler's letter to Euler

Around the same time, on 13 March 1736, Euler wrote to Giovanni Marinoni, an Italian mathematician and engineer who lived in Vienna and was Court Astronomer in the court of Kaiser Leopold I. He introduced the problem as follows (see Figure 6):

> A problem was posed to me about an island in the city of Königsberg, surrounded by a river spanned by seven bridges, and I was asked whether someone could traverse the separate bridges in a connected walk in such a way that each bridge is crossed only once. I was informed that hitherto no-one had demonstrated the possibility of doing this, or shown that it is impossible. This question is so banal, but seemed to me worthy of attention in that geometry, nor algebra, nor even the art of counting was sufficient to solve it. In view of this, it occurred to me to wonder whether it belonged to the geometry of position [*geometriam Situs*], which Leibniz had once so much longed for. And so, after some deliberation, I obtained a simple, yet completely established, rule with whose help one can immediately decide for all examples of this kind, with any number of bridges in any arrangement, whether such a round trip is possible, or not ...

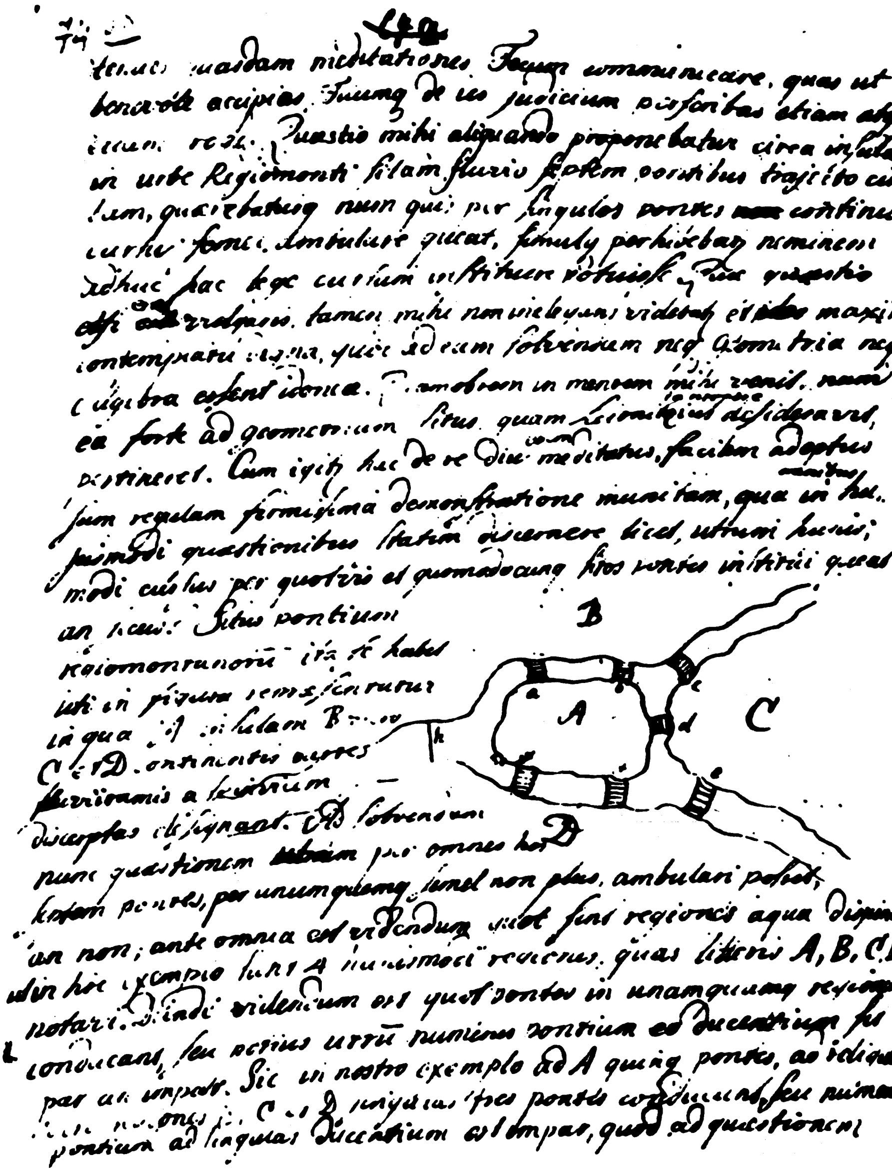

Figure 6. Euler's letter to Marinoni

Euler's 1736 paper

Euler's paper is divided into twenty-one numbered paragraphs, of which the first ascribes the problem to the geometry of position as we saw above, the next eight are devoted to the solution of the Königsberg bridges problem itself, and the remainder are concerned with the general problem. More specifically, paragraphs 2–21 deal with the following topics (see also [**12**]):

Paragraph 2. Euler described the problem of the Königsberg bridges and its generalization: 'whatever be the arrangement and division of the river into branches, and however many bridges there be, can one find out whether or not it is possible to cross each bridge exactly once?'

Paragraph 3. In principle, the original problem could be solved exhaustively by checking all possible paths, but Euler dismissed this as 'laborious' and impossible for configurations with more bridges.

Paragraphs 4–7. The first simplification is to record paths by the land regions rather than bridges. Using the notation in Figure 4, going south from Kneiphof would be notated AB whether one used the Green Bridge or the Blacksmith's Bridge. The final path notation will need to include an adjacent A and B twice; the particular assignment of bridges a and b is irrelevant. A path signified by n letters corresponds to crossing $n-1$ bridges, so a solution to the Königsberg problem requires an eight-letter path with two adjacent A/B pairs, two adjacent A/C pairs, one adjacent A/D pair, etc.

Paragraph 8. What is the relation between the number of bridges connecting a land mass and the number of times the corresponding letter occurs in the path? Euler developed the answer from a simpler example (see Figure 7). If there is an odd number k of bridges, then the letter must appear $(k+1)/2$ times.

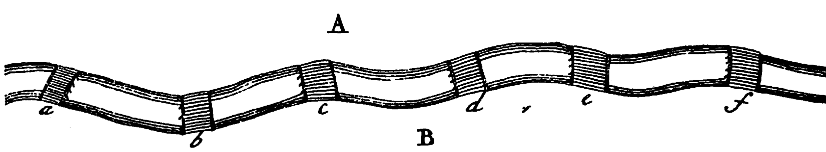

Figure 7. A simple case

Paragraph 9. This is enough to establish the impossibility of the desired Königsberg tour. Since Kneiphof is connected by five bridges, the path must contain three As. Similarly, there must be two Bs, two Cs, and two Ds. In *paragraph 14*, Euler records these data in a table.

region	A	B	C	D
bridges	5	3	3	3
frequency	3	2	2	2

Summing the final row gives nine required letters, but a path using each of the seven bridges exactly once can have only eight letters. Thus there can be no Königsberg tour.

Paragraphs 10–12. Euler continued his analysis from *paragraph 8:* if there is an even number k of bridges connecting a land mass, then the corresponding letter appears $k/2+1$ times if the path begins in that region, and $k/2$ times otherwise.

Paragraphs 13–15. The general problem can now be addressed. To illustrate the method Euler constructed an example with two islands, four rivers, and fifteen bridges (see Figure 8).

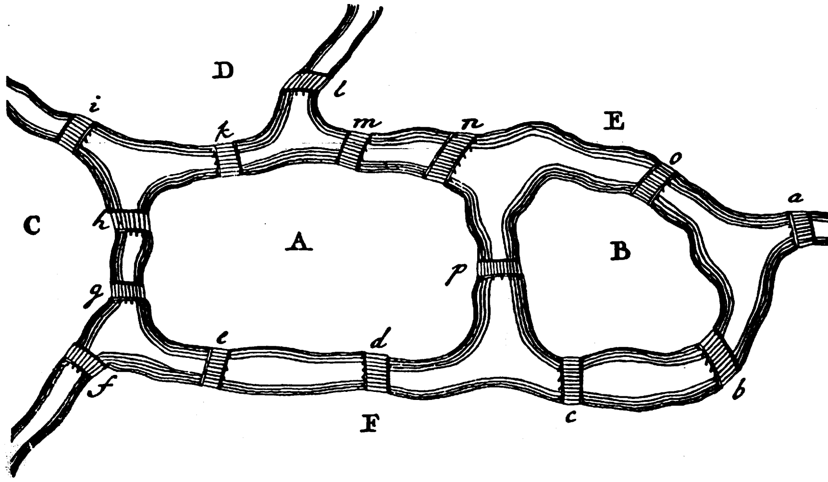

Figure 8. A more complicated example

This system has the following table, where an asterisk indicates a region with an even number of bridges.

region	A^*	B^*	C^*	D	E	F^*
bridges	8	4	4	3	5	6
frequency	4	2	2	2	3	3

The frequencies of the letters in a successful path are determined by the rules for even and odd numbers of bridges, developed above. Since there can be only one initial region, he records $k/2$ for the asterisked regions. If the frequency sum is one less than the required number of letters, there is a path using each bridge exactly once that begins in an asterisked region. If the frequency sum equals the required number of letters, there is a path that begins in an unasterisked region. This latter possibility is the case here: the frequency sum is 16, exactly the number of letters required for a path using 15 bridges. Euler exhibited a particular path, including the bridges:

$$E\,a\,F\,b\,B\,c\,F\,d\,A\,e\,F\,f\,C\,g\,A\,h\,C\,i\,D\,k\,A\,m\,E\,n\,A\,p\,B\,o\,E\,l\,D.$$

Paragraph 16–19. Euler continued with a simpler technique, observing that:

> ... the number of bridges written next to the letters A, B, C, etc. together add up to twice the total number of bridges. The reason for this is that, in the calculation where every bridge leading to a given area is counted, each bridge is counted twice, once for each of the two areas which it joins.

This is the earliest version known of what is now called the *handshaking lemma*. It follows that in the bridge sum, there must be an even number of odd summands.

Paragraph 20. Euler stated his main conclusions:

> If there are more than two areas to which an odd number of bridges lead, then such a journey is impossible.

> If, however, the number of bridges is odd for exactly two areas, then the journey is possible if it starts in either of these two areas.
>
> If, finally, there are no areas to which an odd number of bridges lead, then the required journey can be accomplished starting from any area.

Paragraph 21. Euler concluded by saying:

> When it has been determined that such a journey can be made, one still has to find how it should be arranged. For this I use the following rule: let those pairs of bridges which lead from one area to another be mentally removed, thereby considerably reducing the number of bridges; it is then an easy task to construct the required route across the remaining bridges, and the bridges which have been removed will not significantly alter the route found, as will become clear after a little thought. I do not therefore think it worthwhile to give any further details concerning the finding of the routes.

Note that this final paragraph does not prove the existence of a journey when one is possible, apparently because Euler did not consider it necessary. So Euler provided a rigorous proof only for the first of the three conclusions. The first satisfactory proof of the other two results did not appear until 1871, in a posthumous paper by Carl Hierholzer (see [**1**] and [**4**]).

The modern solution

The approach mentioned in the first section developed through diagram-tracing puzzles discussed by Louis Poinsot [**7**] and others in the early-nineteenth century. The object is to determine whether a figure can be drawn with a single stroke of the pen in such a way that no edge is repeated. Considering the figure to be drawn as a graph, the general conditions in *Paragraph 20* take the following form:

> If there are more than two vertices of odd degree, then such a drawing is impossible.
>
> If, however, exactly two vertices have odd degree, then the drawing is possible if it starts with either of these two vertices.
>
> If, finally, there are no vertices of odd degree, then the required drawing can be accomplished starting from any vertex.

So the 4-vertex graph shown in Figure 2, with one vertex of degree 5 and three vertices of degree 3, cannot be drawn with a single stroke of the pen so that no edge is repeated. In contemporary terminology, we say that this graph is not Eulerian. The arrangement of bridges in Figure 8 can be similarly represented by the graph in Figure 9, with six vertices and fifteen edges. Exactly two vertices (E and D) have odd degree, so there is a drawing that starts at E and ends at D, as we saw above. This is sometimes called an Eulerian trail.

However, it was some time until the connection was made between Euler's work and diagram-tracing puzzles. The 'Königsberg graph' of Figure 2 made its first appearance in W. W. Rouse Ball's *Mathematical Recreations and Problems of Past and Present Times* [**9**] in 1892.

Background information, including English translations of the papers of Euler [**2**] and Hierholzer [**4**], can be found in [**1**]; an English translation of Euler's paper also appears in [**6**].

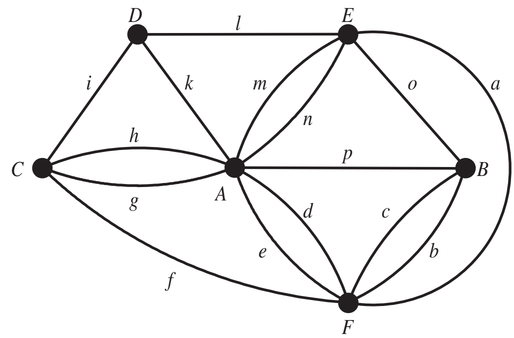

Figure 9. The graph of the bridges in Figure 8

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If I feel unhappy, I do mathematics to become happy. If I am happy, I do mathematics to keep happy. ——Alfréd Rényi

(Quoted in P. Turán, "The Work of Alfréd Rényi," *Matematikai Lapok* **21** (1970) 199–210)

SOLUTIO PROBLEMATIS AD GEOMETRIAM SITUS PERTINENTIS

Commentatio 53 indicis ENESTROEMIANI
Commentarii academiae scientiarum Petropolitanae 8 (1736), 1741, p. 128—140

1. Praeter illam geometriae partem, quae circa quantitates versatur et omni tempore summo studio est exculta, alterius partis etiamnum admodum ignotae primus mentionem fecit LEIBNITZIUS[1]), quam *Geometriam situs* vocavit. Ista pars ab ipso in solo situ determinando situsque proprietatibus eruendis occupata esse statuitur; in quo negotio neque ad quantitates respiciendum neque calculo quantitatum utendum sit. Cuiusmodi autem problemata ad hanc situs geometriam pertineant et quali methodo in iis resolvendis uti oporteat, non satis est definitum. Quamobrem, cum nuper problematis cuiusdam mentio esset facta, quod quidem ad geometriam pertinere videbatur, at ita erat comparatum, ut neque determinationem quantitatum requireret neque solutionem calculi quantitatum ope admitteret, id ad geometriam situs referre haud dubitavi, praesertim quod in eius solutione solus situs in considerationem veniat, calculus vero nullius prorsus sit usus. Methodum ergo meam, quam ad huius generis problemata solvenda inveni, tanquam specimen Geometriae situs hic exponere constitui.

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1) Vide epistolam a G. LEIBNIZ (1646—1716) ad CHR. HUYGENS (1629—1695) scriptam d. 8. Sept. 1679, *LEIBNIZENS Mathematische Schriften*, herausg. von C. I. GERHARDT, Erste Abt., Bd. 2, Berlin 1850, p. 17—20, imprimis p. 19, Beilage p. 20—25. Epistola, qua HUYGENS respondit d. 22. Nov. 1679, invenitur ibidem p. 27. Vide porro G. LEIBNIZ, *De analysi situs*, *LEIBNIZENS Mathematische Schriften*, Zweite Abt., Bd. 1, Halle 1858, p. 178—183. L. G. D.

2. Problema autem hoc, quod mihi satis notum esse perhibebatur, erat sequens: Regiomonti in Borussia esse insulam *A*, *der Kneiphof* dictam, fluviumque eam cingentem in duos dividi ramos, quemadmodum ex figura (Fig. 1) videre licet; ramos vero huius fluvii septem instructos esse pontibus *a*, *b*, *c*, *d*, *e*, *f* et *g*. Circa hos pontes iam ista proponebatur quaestio, num quis cursum ita instituere queat, ut per singulos pontes semel et non plus quam semel transeat. Hocque fieri posse, mihi dictum est, alios negare alios dubitare; neminem vero affirmare. Ego ex hoc mihi sequens maxime generale formavi problema: quaecunque sit fluvii figura et distributio in ramos atque quicunque fuerit numerus pontium, invenire, utrum per singulos pontes semel tantum transiri queat an vero secus.

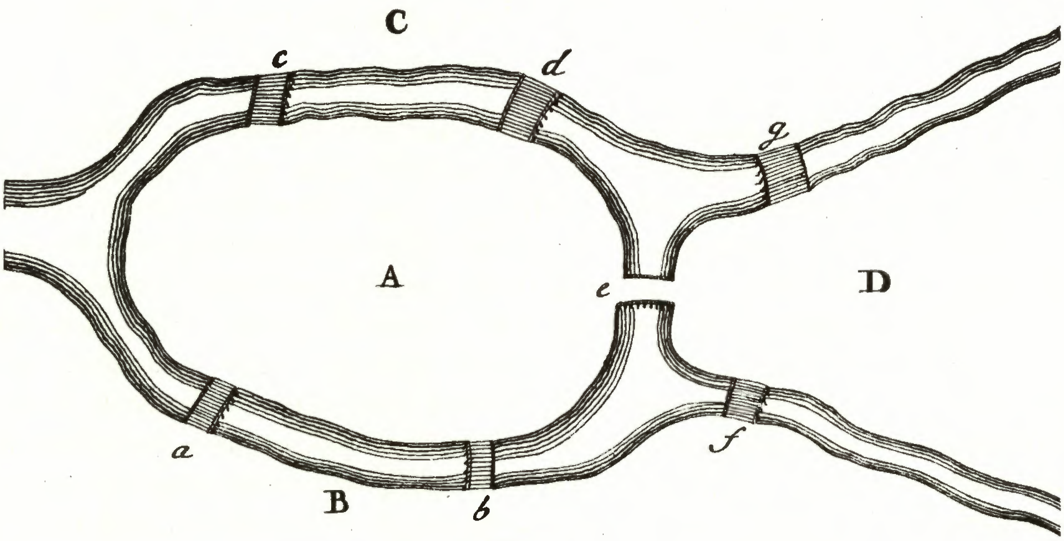

Fig. 1.

3. Quod quidem ad problema Regiomontanum de septem pontibus attinet, id resolvi posset facienda perfecta enumeratione omnium cursuum, qui institui possunt; ex his enim innotesceret, num quis cursus satisfaceret an vero nullus. Hic vero solvendi modus propter tantum combinationum numerum et nimis esset difficilis atque operosus et in aliis quaestionibus de multo pluribus pontibus ne quidem adhiberi posset. Hoc porro modo si operatio ad finem perducatur, multa inveniuntur, quae non erant in quaestione; in quo procul dubio tantae difficultatis causa consistit. Quamobrem missa hac me-

thodo in aliam inquisivi, quae plus non largiatur, quam ostendat, utrum talis cursus institui queat an secus; talem enim methodum multo simpliciorem fore sum suspicatus.

4. Innititur autem tota mea methodus idoneo modo singulos pontium transitus designandi, in quo utor litteris maiusculis A, B, C, D singulis regionibus adscriptis, quae flumine sunt separatae. Ita, si quis ex regione A in regionem B transmigrat per pontem a sive b, hunc transitum denoto litteris AB, quarum prior praebet regionem, ex qua exierat viator, posterior vero dat regionem, in quam pontem transgressus pervenit. Si deinceps viator ex regione B abeat in regionem D per pontem f, hic transitus repraesentabitur litteris BD; duos autem hos transitus successive institutos AB et BD denoto tantum tribus litteris ABD, quia media B designat tam regionem, in quam primo transitu pervenit, quam regionem, ex qua altero transitu exit.

5. Simili modo si viator ex regione D progrediatur in regionem C per pontem g, hos tres transitus successive factos quatuor litteris $ABDC$ denotabo. Ex his enim quatuor litteris $ABDC$ intelligetur viatorem primo in regione A existentem transiisse in regionem B, hinc esse progressum in regionem D ex hacque ultra esse profectum in C; cum vero hae regiones fluviis sint a se invicem separatae, necesse est, ut viator tres pontes transierit. Sic transitus per quatuor pontes successive instituti quinque litteris denotabuntur; et si viator trans quotcunque pontes eat, eius migratio per litterarum numerum, qui unitate est maior quam numerus pontium, denotabitur. Quare transitus per septem pontes ad designandum octo requirit litteras.

6. In hoc designandi modo non respicio, per quos pontes transitus sit factus, sed si idem transitus ex una regione in aliam per plures pontes fieri potest, perinde est, per quemnam transeat, modo in designatam regionem perveniat. Ex quo intelligitur, si cursus per septem figurae pontes ita institui posset, ut per singulos semel ideoque per nullum bis transeatur, hunc cursum octo litteris repraesentari posse easque litteras ita esse debere dispositas, ut immediata litterarum A et B successio bis occurrat, quia sunt duo pontes a et b has regiones A et B iungentes; simili modo successio litterarum A et C quoque debet bis occurrere in illa octo litterarum serie; deinde successio litterarum A et D semel occurret similiterque successio litterarum B et D itemque C et D semel occurrat necesse est.

7. Quaestio ergo huc reducitur, ut ex quatuor litteris A, B, C et D series octo litterarum formetur, in qua omnes illae successiones toties occurrant, quoties est praeceptum. Antequam autem ad talem dispositionem opera adhibeatur, ostendi convenit, utrum tali modo hae litterae disponi queant an non. Si enim demonstrari poterit talem dispositionem omnino fieri non posse, inutilis erit omnis labor, qui ad hoc efficiendum locaretur. Quamobrem regulam investigavi, cuius ope tam pro hac quaestione quam pro omnibus similibus facile discerni queat, num talis litterarum dispositio locum habere queat.

8. Considero ad huiusmodi regulam inveniendam unicam regionem A, in quam quotcunque pontes a, b, c, d etc. conducant (Fig. 2). Horum pontium contemplor primo unicum a, qui ad regionem A ducat; si nunc viator per hunc pontem transeat, vel ante transitum esse debuit in regione A vel post transitum in A perveniet; quare in supra stabilito transitus designandi modo oportet, ut littera A semel occurrat. Si tres pontes, puta a, b, c, in regionem A conducant et viator per omnes tres transeat, tum in designatione eius migrationis littera A bis occurret, sive ex A initio cursum instituerit sive minus. Simili modo si quinque pontes in A conducant, in designatione transitus per eos omnes littera A ter occurrere debet. Atque si numerus pontium fuerit quicunque numerus impar, tum, si is unitate augeatur, eius dimidium dabit, quot vicibus littera A occurrere debeat.

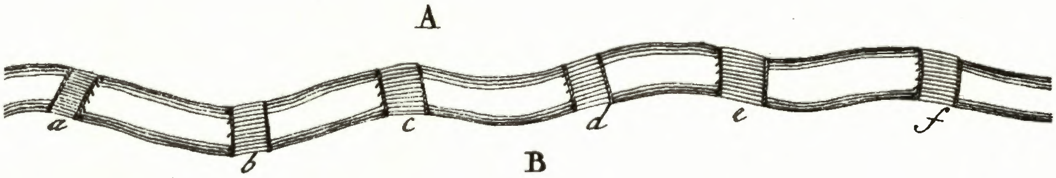

Fig. 2.

9. In casu igitur pontium transeundorum Regiomontano (Fig. 1), quia in insulam A quinque pontes deducunt a, b, c, d, e, necesse est, ut in designatione transitus per hos pontes littera A ter occurrat. Deinde littera B, quia in regionem B tres pontes conducunt, bis debet occurrere similique modo littera D bis debet occurrere atque etiam littera C bis. In serie ergo octo litterarum, quibus transitus per septem pontes deberet designari, littera A ter adesse deberet, litterarum vero B, C et D unaquaeque bis; id quod in serie octo litterarum omnino fieri nequit. Ex quo perspicuum est per septem pontes Regiomontanos talem transitum institui non posse.

10. Simili modo de omni alio casu pontium, si quidem numerus pontium, qui in quamque regionem conducit, fuerit impar, iudicari potest, an per singulos pontes transitus semel fieri queat. Si enim evenit, ut summa omnium vicium, quibus singulae litterae occurrere debent, aequalis sit numero omnium pontium unitate aucto, tum talis transitus fieri potest; sin autem, ut in nostro exemplo accidit, summa omnium vicium maior fuerit numero pontium unitate aucto, tum talis transitus nequaquam institui potest. Regula autem, quam dedi pro numero vicium A ex numero pontium in regionem A deducentium inveniendo, aeque valet, sive omnes pontes ex una regione B, ut in figura (Fig. 2) repraesentatur, ducant sive ex diversis; tantum enim regionem A considero et inquiro, quot vicibus littera A occurrere debeat.

11. Si autem numerus pontium, qui in regionem A conducunt, fuerit par, tum circa transitum per singulos notandum est, utrum initio viator cursum suum ex regione A instituerit an non. Si enim duo pontes in A conducant et viator ex A cursum inceperit, tum littera A bis occurrere debet; semel enim adesse debet ad designandum exitum ex A per alterum pontem et semel quoque ad designandum reditum in A per alterum pontem. Sin autem viator ex alia regione cursum inceperit, tum semel tantum littera A occurret; semel enim posita tam adventum in A quam exitum inde denotabit, ut huiusmodi cursus designare statui.

12. Conducant iam quatuor pontes in regionem A et viator ex A cursum incipiat; tum in designatione totius cursus littera A ter adesse debebit, si quidem per singulos semel transierit. At si ex alia regione ambulare inceperit, tum bis tantum littera A occurret. Si sex pontes ad regionem A conducant, tum littera A, si ex A initium eundi est sumptum, quater occurret, at si non ex A initio exierit viator, tum ter tantum occurrere debebit. Quare generaliter: si numerus pontium fuerit par, tum eius dimidium dat numerum vicium, quibus littera A occurrere debet, si initium non est in regione A sumptum; dimidium vero unitate auctum dabit numerum vicium, quoties littera A occurrere debet, initio cursus in ipsa regione A sumpto.

13. Quia autem in tali cursu nonnisi ex una regione initium fieri potest, ideo ex numero pontium, qui in quamvis regionem deducunt, ita numerum vicium, quoties littera quamque regionem denotans occurrere debet, definio, ut

sumam numeri pontium unitate aucti dimidium, si numerus pontium fuerit impar; ipsius vero numeri pontium medietatem, si fuerit par. Deinde si numerus omnium vicium adaequet numerum pontium unitate auctum, tum transitus desideratus succedit, at initium ex regione, in quam impar pontium numerus ducit, capi debet. Sin autem numerus omnium vicium fuerit unitate minor quam pontium numerus unitate auctus, tum transitus succedet incipiendo ex regione, in quam par pontium numerus ducit, quia hoc modo vicium numerus unitate est augendus.

14. Proposita ergo quacunque aquae pontiumque figura ad investigandum, num quis per singulos semel transire queat, sequenti modo operationem instituo. Primo singulas regiones aqua a se invicem diremptas litteris *A*, *B*, *C* etc. designo. Secundo sumo omnium pontium numerum eumque unitate augeo atque sequenti operationi praefigo. Tertio singulis litteris *A*, *B*, *C* etc. sibi subscriptis cuilibet adscribo numerum pontium ad eam regionem deducentium. Quarto eas litteras, quae pares adscriptos habent numeros, signo asterisco. Quinto singulorum horum numerorum parium dimidia adiicio, imparium vero unitate auctorum dimidia ipsis adscribo. Sexto hos numeros ultimo scriptos in unam summam coniicio; quae summa si vel unitate minor fuerit vel aequalis numero supra praefixo, qui est numerus pontium unitate auctus, tum concludo transitum desideratum perfici posse. Hoc vero est tenendum, si summa inventa fuerit unitate minor quam numerus supra positus, tum initium ambulationis ex regione asterisco notata fieri debere; contra vero ex regione non signata, si summa fuerit aequalis numero praescripto. Ita ergo pro casu Regiomontano operationem instituo, ut sequitur:

Numerus pontium 7, habetur ergo 8.

	Pontes	
A,	5	3
B,	3	2
C,	3	2
D,	3	2

Quia ergo plus prodit quam 8, huiusmodi transitus nequaquam fieri potest.

15. Sint duae insulae A et B aqua circumdatae, qua cum aqua communicent quatuor fluvii, quemadmodum figura (Fig. 3) repraesentat. Traiecto porro sint super aquam insulas circumdantem et fluvios quindecim pontes a, b, c, d etc. et quaeritur, num quis cursum ita instituere queat, ut per

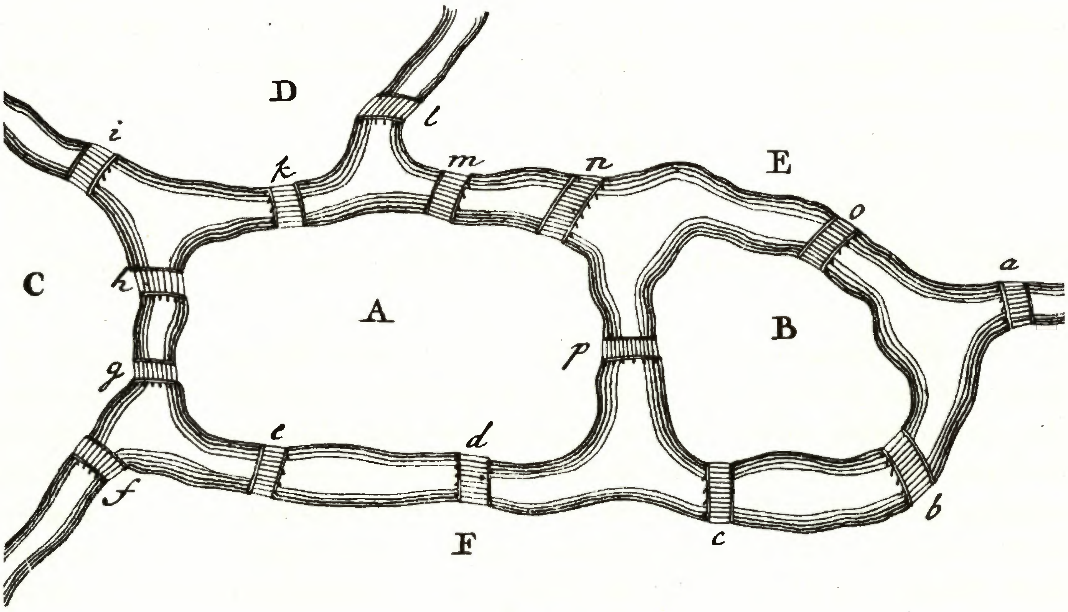

Fig. 3.

omnes pontes transeat, per nullum autem plus quam semel. Designo ergo primum omnes regiones, quae aqua a se invicem sunt separatae, litteris A, B, C, D, E, F, cuiusmodi ergo sunt sex regiones. Dein numerum pontium 15 unitate augeo et summam 16 sequenti operationi praefigo.

		16
A^*,	8	4
B^*,	4	2
C^*,	4	2
D ,	3	2
E ,	5	3
F^*,	6	3
		16

Tertio litteras A, B, C etc. sibi invicem subscribo et ad quamque numerum pontium, qui in eam regionem ducunt, pono, ut ad A octo ducunt pontes, ad B quatuor etc. Quarto litteras, quae pares adiunctos habent numeros, asterisco noto. Quinto in tertiam columnam scribo parium numerorum dimidia, impares vero unitate augeo et semisses appono. Sexto tertiae columnae numeros invicem addo et obtineo summam 16; quae cum aequalis sit numero supra posito 16, sequitur transitum desiderato modo fieri posse, si modo cursus vel ex regione D vel E incipiatur, quippe quae non sunt asterisco notatae. Cursus autem ita fieri poterit

$$EaFbBcFdAeFfCgAhCiDkAmEnApBoElD,$$

ubi inter litteras maiusculas pontes simul collocavi, per quos fit transitus.

16. Hac igitur ratione facile erit in casu quam maxime composito iudicare, utrum transitus per omnes pontes semel tantum fieri queat an non. Hoc tamen adhuc multo faciliorem tradam modum idem dignoscendi, qui ex hoc ipso modo non difficulter eruetur, postquam sequentes observationes in medium protulero. Primo autem observo omnes numeros pontium singulis litteris A, B, C etc. adscriptos simul sumptos duplo maiores esse toto pontium numero. Huius rei ratio est, quod in hoc computo, quo pontes omnes in datam regionem ducentes numerantur, quilibet pons bis numeretur; refertur enim quisque pons ad utramque regionem, quas iungit.

17. Sequitur ergo ex hac observatione summam omnium pontium, qui in singulas regiones conducunt, esse numerum parem, quia eius dimidium pontium numero aequatur. Fieri ergo non potest, ut inter numeros pontium in quamlibet regionem ducentium unicus sit impar; neque etiam, ut tres sint impares, neque quinque etc. Quare si qui pontium numeri litteris A, B, C etc. adscripti sunt impares, necesse est, ut eorum numerus sit par; ita in exemplo Regiomontano quatuor erant pontium numeri impares litteris regionum A, B, C, D adscripti, uti ex § 14 videre licet; atque in exemplo praecedente, § 15, duo tantum sunt numeri impares, litteris D et E adscripti.

18. Cum summa omnium numerorum litteris A, B, C etc. adiunctorum aequet duplum pontium numerum, manifestum est illam summam binario auctam et per 2 divisam dare numerum operationi praefixum. Si igitur

omnes numeri litteris A, B, C, D etc. adscripti fuerint pares et eorum singulorum medietates capiantur ad numeros tertiae columnae obtinendos, erit horum numerorum summa unitate minor quam numerus praefixus. Quamobrem his casibus semper transitus per omnes pontes fieri potest. In quacunque enim regione cursus incipiatur, ea habebit pontes numero pares ad se conducentes, uti requiritur. Sic in exemplo Regiomontano fieri potest, ut quis per omnes pontes bis transgrediatur; quilibet enim pons quasi in duos erit divisus numerusque pontium in quamvis regionem ducentium erit par.

19. Praeterea, si duo tantum numeri litteris A, B, C etc. adscripti fuerint impares, reliqui vero omnes pares, tum semper desideratus transitus succedet, si modo cursus ex regione, ad quam pontium impar numerus tendit, incipiatur. Si enim pares numeri bisecentur atque etiam impares unitate aucti, uti praeceptum est, summa harum medietatum unitate erit maior quam numerus pontium ideoque aequalis ipsi numero praefixo.

Ex hocque porro perspicitur, si quatuor vel sex vel octo etc. fuerint numeri impares in secunda columna, tum summam numerorum tertiae columnae maiorem fore numero praefixo eumque excedere vel unitate vel binario vel ternario etc. et idcirco transitus fieri nequit.

20. Casu ergo quocunque proposito statim facillime poterit cognosci, utrum transitus per omnes pontes semel institui queat an non, ope huius regulae:

Si fuerint plures duabus regiones, ad quas ducentium pontium numerus est impar, tum certo affirmari potest talem transitum non dari.

Si autem ad duas tantum regiones ducentium pontium numerus est impar, tunc transitus fieri poterit, si modo cursus in altera harum regionum incipiatur.

Si denique nulla omnino fuerit regio, ad quam pontes numero impares conducant, tum transitus desiderato modo institui poterit, in quacunque regione ambulandi initium ponatur.

Hac igitur data regula problemati proposito plenissime satisfit.

21. Quando autem inventum fuerit talem transitum institui posse, quaestio superest, quomodo cursus sit dirigendus. Pro hoc sequenti utor regula: tollantur cogitatione, quoties fieri potest, bini pontes, qui ex una regione in aliam ducunt, quo pacto pontium numerus vehementer plerumque diminuetur; tum quaeratur, quod facile fiet, cursus desideratus per pontes reliquos; quo invento pontes cogitatione sublati hunc ipsum cursum non multum turbabunt, id quod paululum attendenti statim patebit; neque opus esse iudico plura ad cursus reipsa formandos praecipere.

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