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The difficult birth of stochastics: Jacob Bernoulli's *Ars Conjectandi* (1713)

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Abstract

Jacob Bernoulli (1654–1705) did most of his research on the mathematics of uncertainty – or stochastics, as he came to call it – between 1684 and 1690. However, the *Ars Conjectandi*, in which he presented his insights (including the fundamental "Law of Large Numbers"), was printed only in 1713, eight years after his death. The paper studies the sources and the development of Bernoulli's ideas on probability, the reasons behind the delay in publishing and the circumstances under which his masterpiece eventually reached the public.

Zusammenfassung

Jacob Bernoulli (1654–1705) entwickelte seine Ideen zur mathematischen Erforschung des Ungewissen in den Jahren 1684–1690. In seinem Hauptwerk *Ars Conjectandi* stellte er seine neuen Einsichten (darunter das fundamentale "Gesetz der grossen Zahl") vor und prägte den Begriff der Stochastik. Das Buch wurde jedoch erst 1713, acht Jahre nach seinem Tod, gedruckt. Die vorliegende Arbeit studiert die Anstöße zu Bernoullis Beschäftigung mit Wahrscheinlichkeiten, die Entwicklung seiner Auffassungen, die Gründe, die zu der verzögerten Publikation führten, und die Umstände, unter denen das Werk schließlich doch das Licht der Welt erblickte.

MSC: 01A45; 01A50; 60-03

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When Jacob Bernoulli died on August 16th, 1705, at the age of only 50, his *chef d'oeuvre* did not exist as a printed book: this only came out of the Thurneysen brothers' presses eight years later.[1] But it had

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[1] Bernoulli, Jacob (1713), *Ars Conjectandi. Opus posthumum* (see Figure 1). This has been edited with extensive documentation by B.L. van der Waerden, J. Henny and K. Kohli in Jac. B. *Werke* 3, pp. 107–286. There is an English translation with ample historical commentary by E.D. Sylla (2006). Unavoidably, the present paper considerably overlaps with both these editions; however, its

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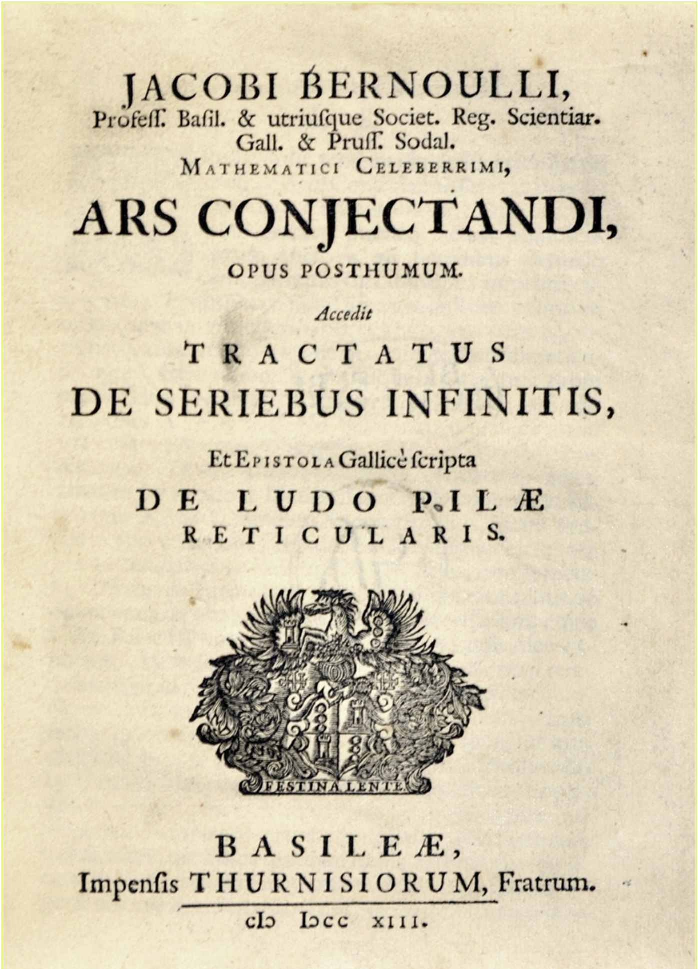

Figure 1. Jacob Bernoulli, *Ars Conjectandi*, Basel 1713, title page.

been there for about fifteen years as a more or less coherent manuscript which Bernoulli – a slow and painstaking author – seems to have thought almost but not quite publishable. And, as so many other ideas at the time did, it floated around as part of mathematical folklore, rumoured to exist in a small informal network of scientists. Actually the few colleagues with whom Bernoulli was in contact had often ignored or misunderstood the hints he dropped about his research on "stochastics", as he called it, but Leibniz, l'Hôpital, Varignon, Bernoulli's students Jacob Hermann and Nicolaus Bernoulli, and even his own brother Johann knew he had worked on such a project. All the same, it is no exaggeration to say that in 1705 the chances for the brainchild's survival were slim: the complex of ideas that many take to be the first in-depth investigation of uncertainty might well have been lost.

The main point of this paper is telling the story behind the *Ars Conjectandi*'s belated publication. But first let us have a look at how it was conceived in Jacob Bernoulli's mind.

main focus is on the interaction between the biographical and philosophical background for the development of Bernoulli's novel ideas on chance, their (largely missing) impact during his lifetime and their reception in the decade after his death.

Among the copious secondary literature on Bernoulli's seminal work, let us mention Hacking (1971), Shafer (1996) (with a careful analysis of concepts and perspectives), Schneider (2005) (the same volume also contains sections on de Moivre's and Laplace's contributions to probability theory), Collani (2006) (presenting a "dissident" view of Bernoulli's ideas on uncertainty) and Seneta (2013).

1. An unplanned conception: where did the ideas in the *Ars Conjectandi* come from?

Bernoulli trained as a theologian: he was ordained as a minister of the Reformed church at Basel in 1676 and worked as a private tutor and chaplain at Geneva and in Southwest France for several years.

The intellectual diary in which he noted, for all his adult life, his "theological and philosophical meditations, notes and remarks" tells us about his early interests.[2] They include what was then called natural philosophy, i.e., speculation in theoretical physics and particularly cosmology; and they also show a vivid awareness of the workings of the mind in perceiving the world, forming concepts, judging truth and making decisions. In the early *Meditationes* we find questions that may be paraphrased as: "Should we keep our hats on in church in countries where this is a sign of respect?", "How can we know that we read Scripture the right way?" or "Do we rely on human testimony in believing that the Bible tells the truth?".[3] Bernoulli had learned to ponder arguments carefully and evaluate them in an ordered fashion. An important source for these epistemological speculations, which he quotes in several places, is the "Logique de Port-Royal" by the progressive Catholic philosophers Antoine Arnauld and Pierre Nicole.[4] When Bernoulli later compiled his rules for achieving insight and coming to decisions under a regime of incomplete information, he modelled the title of his work on that of Arnauld's textbook of logic: the art of thinking, *Ars Cogitandi*, which dealt with the deduction of solid truth, became an art of reasoning under conditions of uncertainty, *Ars Conjectandi*.

The next step in Bernoulli's development was marked by Cartesianism – then the advanced school of thinking about the world, still controversial in conservative countries such as Switzerland but very much in fashion in France and the Netherlands. Bernoulli avidly studied the modern physics and started to publish the first papers of his own in the early 1680s.

Among the books he bought at Leiden in 1682, one finds a collection of treatises by the Dutch mathematician Frans van Schooten.[5] Most of these deal with elementary Cartesian geometry, a subject that Bernoulli studied very carefully and enriched substantially by his later work. However, the most interesting part in our context is the appendix in which Schooten published his Latin translation of a little tract by Christiaan Huygens, *Van Rekeningh in spelen van geluck*.[6] On these 14 pages, Huygens presents a series of simple propositions which establish rules for calculation of what he calls the "value of expectation" in games of chance; and at the end, he proposes five – elementary but not easy – exercise problems.

There is no need to go into detail about the concepts of hazard or expectation involved here or in earlier tracts on gambling, since these have often been analysed by historians of probability theory.[7] Let it suffice to say that Huygens' paper gives some rather simple tools for calculating ratios of chances and a challenging set of questions on which to cut one's teeth.

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2 *Meditationes, Annotationes, Animadversiones Theologicae & Philosophicae, a me JB. concinnatae & collectae ab anno 1677*. This diary is conserved at the Manuscript Department of the Basel University Library (Ms UB Basel L Ia 3). The editors of Jacob Bernoulli's *Werke* decided to distribute its 286 articles among the individual volumes according to their subject matter; most but not all of them have been edited by now. They will be quoted as Med. ### (*Werke* #, pp. ###).

3 Med. III, *An in China aliisque locis tecto aut aperto capite sacra peragenda sint?*; "Unde credis, te recte legere quod legis?" (in Med. XV); Med. XV, *An divinitas Scripturae pendeat vel ultimo resolvatur in testimonium humanum?* These early *Meditationes* on theological and philosophical questions are still unpublished.

4 Arnauld and Nicole (1682).

5 Schooten (1657).

6 Huygens (1657), *De ratiociniis in ludo aleae*: Schooten (1657), pp. 521–534. The text is reprinted, along with Jacob Bernoulli's solutions and remarks, as Part I of *Ars Conjectandi* (cf. note 1), pp. 1–71.

7 See, e.g., Hacking (2006), Stigler (1986) or Hald (2003).

This is exactly what Jacob Bernoulli did soon after his definitive return to Basel in 1684: the *Meditationes* articles LXIII–LXVII present his solutions of Huygens' five problems, and the next three deal with similar problems in card games.[8]

About a year later, ideas for "external" applications – outside the gambling room – begin to surface. The first one is about a marriage contract[9]: How should the inheritances both partners expect from their parents be split when one of them dies? Bernoulli first calculates a solution which relies on the hypothesis that any order of demise among the persons involved is equally probable. However, then he hesitates and notes that the bride is more likely to survive her father and father-in-law. He tries to take account of this by considering as an additional variable of his model the number of diseases that could kill each one. But then, falling into doubt again, he tersely remarks: "no general solution by variables can be given". Such assumptions on accidental causes of death are more or less arbitrary and can certainly not be "precisely and scientifically" ascertained. In these "civic and moral" issues the true state of facts can never be determined accurately, but only in a probable way, not "*a priori*, from its cause", but only "*a posteriori*, from its outcome", by observing what happens in a multitude of similar cases (cf. Figure 2).[10]

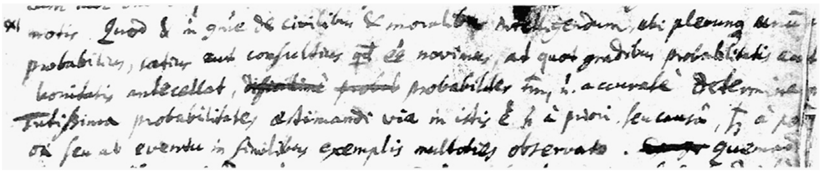


Figure 2. Med. LXXVII, *Meditationes*, manuscript p. 104 (excerpt).

This note from the winter 1685/86 precisely marks the birth of stochastics. The important point is that the word "probability" used here does *not* refer to the ratio of cases in some definite calculating scheme, i.e. to the quantity that earlier studies had focused on computing. Instead it refers to the manner in which *the model itself* can be approved, i.e. shown to yield a fair solution, to its verification by observed reality.

It is exactly at this point that Bernoulli later added a note in the margin of his diary which says: "Indeed I can deviate less in proportion if I observe more often than rarely. I prove this in the addendum" (cf. Figure 3).[11]

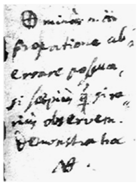


Figure 3. Med. LXXVII, *Meditationes*, manuscript p. 104 (excerpt: marginal note).

As anyone who is at home in probability theory will at once see, this note points to the central theorem of the *Ars Conjectandi*, Bernoulli's "Law of Large Numbers".

[8] Jac. B. *Werke* 3, pp. 21–41.

[9] Med. LXXVII (*Werke* 3, pp. 42–48).

[10] "Quod & in genere de civilibus & moralibus intelligendum, ubi plerunque unum [altero] probabilius, satius aut consultius quidem esse novimus, at quot gradibus probabilitatis aut bonitatis antecellat, probabiliter tantum, non accurate determina[mus]. Tutissima probabilitates aestimandi via in istis est non a priori, seu causa, sed a po[steri]ori seu ab eventu in similibus exemplis multoties observato." (*Werke* 3, p. 46).

[11] "Minus enim in proportione aberrare possum, si saepius quam si rarius observem. Demonstro hoc NB." (*Werke* 3, p. 47, marginal note).

The remainder of Med. LXXVII goes on to propose a host of other "civic" questions that could be investigated in the same way: How many humans will be born or die in some town within a year? How likely is it that an epidemic will return soon? How is the credibility of a witness affected by his previous record of truthfulness?

There is no need to continue further along the series of 23 *Meditationes* classified as "probabilistic" by the editors of Bernoulli's works[12]: with this note the central themes of his stochastics are set. He will indeed suggest many more issues that might be amenable to his new analysis: insurance contracts, elections, measurements, missing person cases, speculation on future crops, detention pending trial, the utility of medicaments, even the ascent to eternal bliss (here Bernoulli cites "Pascal's bet"). Only very few of these applications were realised during Bernoulli's lifetime or, indeed, before the 20th century; but the suggestions are actually all there in his notebook.

The proof of validity announced in Bernoulli's marginal note took some time in coming. A first, tentative sketch is noted after Med. CXXXIII, the complete version – very similar to the later publication – comes after Med. CLI, which can be dated with some certainty to 1689. It starts with the statement of the theorem: "It is possible to perform so many observations that it is more probable than any given probability that the number of games won by either player falls within some given limits, however narrow, than outside them" (cf. Figure 4).[13]

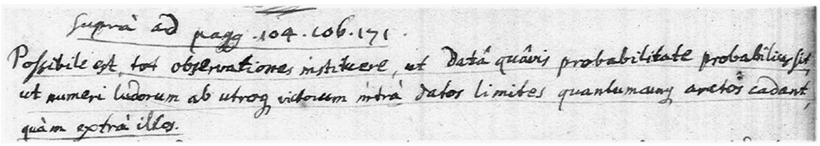

Supra ad pagg. 104. 106. 171.
Possibile est, tot observationes instituere, ut datâ quâvis probabilitate probabilius sit, ut numeri ludorum ab utroq victorum intrâ datos limites quantumcunq arctos cadant, quàm extrà illos.

Figure 4. Med. CLIa, *Meditationes*, manuscript p. 185 (excerpt).

The proof, which fills six pages of small script, ends with the frequently cited proud remark: "I prize this discovery more highly than if I had given the very quadrature of the circle: for even if this were discovered, it would be of little use."[14]

Thus most of the mathematics that is in the *Ars Conjectandi* was in place before 1690, and Jacob Bernoulli was conscious of its novelty, its intrinsic value and the range of potential applications. So the question cannot be sidestepped: why ever did he not publish it?

2. A long gestation: why wasn't the *Ars Conjectandi* published twenty years earlier?

It is not quite true that nothing of Bernoulli's ideas about the investigation of uncertainty seeped out in public. Actually some traces can already be found in the papers he submitted to show his qualifications for an academic job at the Basel university during the mid-1680s. A first sentence which shows Bernoulli's continuing pursuit of epistemological questions is in a collection of 100 short claims offered for debate: "Nothing is in our understanding which has not previously been in our senses."[15] This empiricist maxim is sometimes attributed to John Locke, but was in fact widely spread much earlier; Bernoulli possibly quotes it from Arnauld's *Ars Cogitandi*. Even if no reference is made to chance or probability, the sentence gives a good background for basing opinions and decisions under uncertainty not on preconceived models, but on observation.

12 *Werke* 3, pp. 21–89.

13 "Possibile est, tot observationes instituere, ut data quavis probabilitate probabilius sit, ut numeri ludorum ab utroque victorum intra datos limites quantumcunque arctos cadant, quam extra illos" (*Werke* 3, p. 76).

14 "Hoc inventum pluris facio quam si ipsam circuli quadraturam dedissem, quae si maxime reperiretur, exigui usus esset" (*Werke* 3, p. 88).

15 "... nihil est in intellectu, quod non prius fuerit in sensu": Jac. B. Op. VII (1684), Theses Logicae, 3 (*Werke* 1, p. 242).

The next application paper is titled *Parallelismus ratiocinii logici et algebraici*. Its overall aim is the promotion of mathematical forms of argument in all sciences: as quoted from Malebranche, "algebra is the true logic, useful for discovering truth and giving all the extension to the mind of which it is capable".[16] Accordingly, mathematics dominates all other disciplines by being self-sufficient whereas they are dependent on it. He who has acquired a mathematical faculty of judgement will be able to determine instantly what can or cannot be asserted, known or done about any topic at all. Whereas all other sciences can discourse even on the most certain matters only probabilistically, the primacy of mathematics is evident because it gauges even questions affected by hazard and chance in a definitive and most reliable way.[17] Here Bernoulli adduces as examples a gambling problem that will also appear in the *Ars Conjectandi* and a question on marriage contracts very similar to the one discussed above.

These are almost the last public appearances of questions involving chance in Bernoulli's work during his lifetime; and to call them public means using the term very broadly. It is not known whether these specific points were taken up at all at the disputations in question; and the fact that the papers proposing them seldom reached out beyond the local academic community is attested by the difficulties the editor of Bernoulli's collected works, Gabriel Cramer, later faced in locating copies.

Meanwhile Jacob Bernoulli was fully occupied with other tasks: In 1683 he had begun to teach experimental physics to university students; in the following years up to 1691 he started a family, obtained the coveted chair of mathematics, fostered his first master students and got involved in university politics.

His teaching obligations inevitably centred on "mainstream mathematics", i.e., elementary algebra and geometry. Bernoulli also took up the latter in his research, at first in its Cartesian flavour, on which he produced a great many notes in his diary and in journals. From 1687, when he was one of the first to study and understand Leibniz's enigmatic papers, he concentrated on investigating plane curves by techniques of what came to be called differential geometry. This resulted in a host of 25 papers published in eight years – several of these became landmarks on their own – and in spontaneous applause from Leibniz, who told the Bernoulli brothers in 1694 that the new methods were theirs as much as his. But this success also brought about a rift between Jacob and his younger brother Johann, whom he had taught, when Johann sold their new insights for a pension to the noble French mathematician de l'Hôpital, who promptly published the first ever textbook of differential calculus.[18] In 1695 Johann, who had no chance of an academic position in Switzerland, took a chair at Groningen in the Netherlands and started issuing challenges to the mathematical world. The brothers got embroiled in a dispute that lasted for many years, absorbing much of their capacity and damaging their scientific reputation.

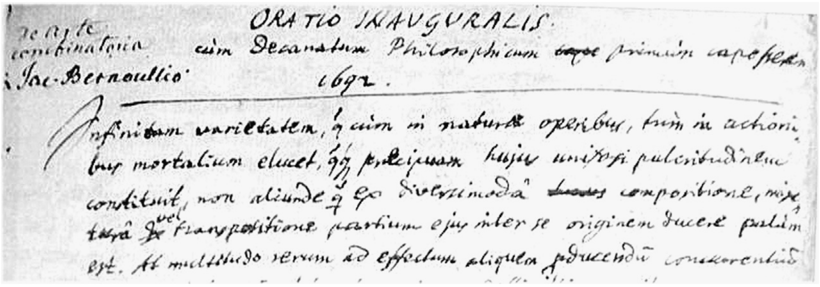
Moreover, Jacob's health had become precarious; he never quite recovered from an illness that had beset him in 1692, and a chronic affliction of his joints increasingly incapacitated him. So in his early forties Jacob Bernoulli, while being acknowledged as one of Europe's foremost mathematicians, had headaches on several fronts: very few people in his provincial hometown were able to appreciate his work, most of his peers abroad took sides with his more energetic brother, and he felt his own creativity declining.

There are very few traces in that decade that Bernoulli's manuscript book, the later *Ars Conjectandi*, still existed, presumably confined to a deep drawer. When he was called to officiate as dean of the philosophical faculty for 1692/1693, he had to give an inaugural speech on some subject chosen from his discipline with a view to public appeal – never a simple task for a mathematician. The topic he chose was the theory of permutations and combinations; and the *Oratio de Arte Combinatoria* (cf. Figure 5) he read on July 5th,

[16] "Algebra est vera Logica, ad detegendam veritatem, omnemque menti, quantae capax est, extensionem dandam, utilis": Jac. B. Op. XVII (1685), Thesis 17 (*Werke* 1, p. 267).

[17] Jac. B. Op. XVII, Theses Miscellaneae, 16–21 (*Werke* 1, pp. 269–270).

[18] l'Hôpital (1696), *Analyse des infiniment petits*.


De Arte combinatoria Jac. Bernoullio

ORATIO INAUGURALIS. cum Decanatum Philosophicum primum caperet 1692.

Infinitam varietatem, quae cum in naturae operibus, tum in actionibus mortalium elucet, quaeque praecipuam hujus universi pulcritudinem constituit, non aliunde quam ex diversimoda compositione, mixtura vel transpositione partium ejus inter se originem ducere palam est. At multitudo rerum ad effectum aliquem producendum concurrentium

Figure 5. *Oratio Inauguralis De Arte Combinatoria*, manuscript p. 1 (excerpt).

1692, in the ceremonial hall of Basel University is almost literally identical with the beginning of Part II of the later book (except for the selection of simpler examples).[19]

Later that year, Jacob's brother must have dropped some hint about his main theorem and the pride he took in it to their colleagues in Paris. On December 8th, 1692, l'Hôpital wrote to Johann Bernoulli: "Ask [your brother] also about what that proposition in his book[!] *de arte conjecturandi*[!] is of which he appreciated the discovery as highly as of the quadrature of the circle".[20] However, Jacob could apparently not be drawn out, and the marquis did not insist.

In the first years of the 18th century, there were again a few rather half-hearted attempts on Bernoulli's part to resume his book project on stochastics. Two conjectures have been advanced about the direction in which he would have expanded the available material if he had had more time and energy left.

On the one hand, Jacob Bernoulli was a perfectionist. It is no accident that the device he adopted for his very first publication reads "What lies so deep comes to light slowly" (cf. Figure 6).[21]

Figure 6. Jac. B. Op. I, title page (excerpt: device and motto).

His best research was developed and reworked in his diary for years, and the precise formulation would take still more time. Bernoulli set great store by meticulous and elegant writing; in fact, he also wrote poems in several languages that show him as an accomplished stylist.

There is no question he would have laboured long hours in order to explain his innovative ideas in a clear and rhetorically attractive form. The introductory sections of Parts II and IV of the *Ars Conjectandi* show the pains taken in order to achieve such an optimal presentation; no doubt Bernoulli planned to devise an equally well-written introduction and recapitulation for the entire book.

[19] *De Arte Combinatoria Oratio Inauguralis*, conserved at the Manuscript Department of the Basel University Library (Ms UB Basel L Ia 749, 1), has been edited in Jac. B. *Werke* 3, pp. 98–106.

[20] "Demandez lui aussi quelle est cette proposition qui est dans son livre de arte conjecturandi dont il estimoit autant la decouverte que la quadrature du cercle." (*Der Briefwechsel von Johann Bernoulli*, vol. 1, p. 160).

[21] "Tarde eruuntur, quae tam alte jacent": Jac. B. Op. I (1681), *Neu-erfundene Anleitung...*, title page (*Werke* 1, p. 135).

It must have been in the course of this editorial work that Bernoulli lit upon the neologism to characterise his new methods that has become very prominent in the 20th century. In Chapter 2 of Part IV one reads: "*To conjecture* some thing means to measure its probability: so we define the *Art of Conjecturing* or *Stochastics* to be the art of measuring as exactly as possible the probabilities of things..."[22] Now the word field of the Latin verb *coniicere* – to throw together, to fling at something, to infer or guess – had been used in the universe of discourse on uncertainty and chance before: the visual metaphor behind it is shooting at a target. On the other hand, the Greek word for the target (στόχος) and the derived verb for aiming at a target (στοχάζεσθαι) had been rarely used in this metaphorical sense. But there is one very specific instance in the Platonic dialogue *Philebos*: "If arithmetic, measurement and weighing were taken away from any art, that which remains would indeed be slight.... All that is left would be guesswork, improvement of the senses by experience and practice, making use of certain skills in hitting the mark (στοχαστική): many take these to be an art by itself, which can be strengthened by exercise and effort".[23] It is characteristic for the scholarly Jacob Bernoulli with his philosophical leanings that he unearthed this passage in order to give his brainchild a classical ancestry – and one that describes very precisely its function as a complement for other, better accredited activities of the mind.

The only documented use of the word *stochastice* in Bernoulli's handwriting is on the initial page of a manuscript in which he collected some notes from the *Meditationes* in view of a publication that he knew would be posthumous, as the title shows (cf. Figure 7).[24]

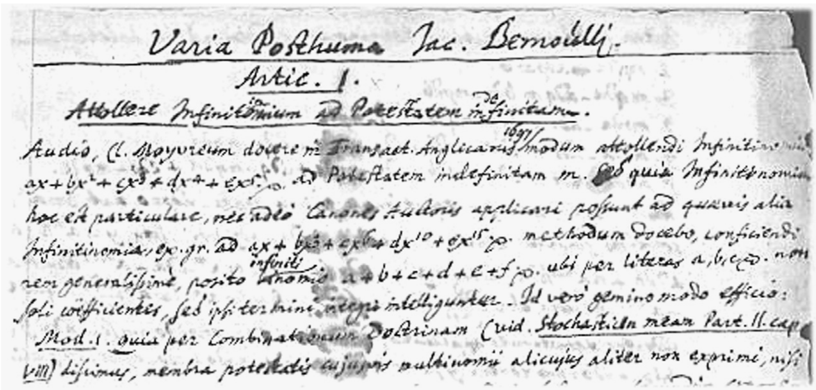


Figure 7. *Varia Posthuma*, manuscript p. 1 (excerpt).

The reference to "my Stochastic, part II, chapter 8" for the theory of combinations which he is applying to a calculation with power series shows that at least this part of the later book was by then available in a final revision.

Let us return to the reasons why Bernoulli, around 1702, considered the *Ars Conjectandi* almost, but not yet quite publishable. Besides the careful formulation of his new and subtle ideas, there was another crucial

22 "*Conjicere* rem aliquam est metiri illius probabilitatem: ideoque *Ars conjectandi* sive *Stochastice* nobis definitur ars metiendi quam fieri potest exactissime probabilitates rerum...": Jacob Bernoulli, *Ars Conjectandi* (1713), p. 213 (*Werke* 3, p. 241).

23 "Οἷον πασῶν που τεχνῶν ἄν τις ἀριθμητικὴν χωρίζῃ καὶ μετρητικὴν καὶ στατικήν, ὡς ἔπος εἰπεῖν φαῦλον τὸ καταλειπόμενον ἑκάστης ἂν γίγνοιτο ... Τὸ γοῦν μετὰ ταῦτ᾽ εἰκάζειν λείποιτ᾽ ἂν καὶ τὰς αἰσθήσεις καταμελετᾶν ἐμπειρίᾳ καί τινι τριβῇ, ταῖς τῆς στοχαστικῆς προσχρωμένους δυνάμεσιν ἃς πολλοὶ τέχνας ἐπονομάζουσι, μελέτῃ καὶ πόνῳ τὴν ῥώμην ἀπειργασμένας" (Platon, *Philebos*, 55e; the paraphrase of Jowett's translation given in the text is intended to fit Bernoulli's use).

24 The *Varia Posthuma* collection of 32 excerpts from the *Meditationes* intended for publication was begun by Jacob Bernoulli himself – thus the title is in his own hand! – and continued by Jacob Hermann, then by Nicolaus I Bernoulli in the context of its edition in Jacob's *Opera*. The manuscript is conserved at the Basel University Library (Ms UB Basel L Ia 1).

upgrading that he must have had in mind. He had to justify his conceptual broadening of what had until then been a collection of rules of thumb for gamblers and to persuade the scientific world that the path on which he had gone the first steps was well worth pursuing. But for this he also needed some more evidence that the applications he imagined were feasible and useful.

In this respect Bernoulli's stochastics was indeed premature: there simply was no foundation of empirical data in his time on which, let's say, a rational theory of the efficiency of medical treatment or of commercial insurance could have been built. In the one area where at least some compilations existed, Jacob Bernoulli's fruitless attempts to get hold of mortality tables of whatever scope and quality bears eloquent witness to the difficulties the nascent science of statistics faced.

In the 1666 *Journal des Savants* Bernoulli had read a review of John Graunt's *Natural and Political Observations made upon the Bills of Mortality*,[25] the first survey of population dynamics ever published; however, he seems never to have seen the book itself. To underpin his study of life annuities, Bernoulli urgently hunted for a printed memorandum by the Dutch civil servant Johan de Witt[26] that contained an extensive mortality table, but his appeals to Leibniz to borrow his copy came to nought. And Halley's paper on the demography of Breslau in the *Philosophical Transactions*[27] does not even seem to have come to his notice. So Bernoulli's intent to prop his favourite brainchild up at least by one meticulously elaborated application failed by an almost complete lack of data.

We can only speculate what prospects Jacob Bernoulli could have opened up and whether his view of stochastics would have been taken up sooner and more broadly, with greater enthusiasm and readier success, if he had lived to perfect his plans.

For all Jacob Bernoulli's unwillingness to discuss his incomplete, imperfect work, some colleagues at least knew – by indiscretions originating with his brother – that it existed. In April 1703, Leibniz asked him in the postscript of a letter: "I have heard that a method for estimating probabilities, which I appreciate a lot, has been greatly enhanced by you. I'd like somebody to treat various kinds of games, where there are beautiful instances of this theory".[28] Bernoulli replies with characteristic moroseness: "I should very much like to know from whom you heard that I am working on a method for estimating probabilities. It is true that several years ago I delighted in speculations of that kind, and I hardly think anybody has given more thought to them. I also had in mind writing a treatise on that subject, but I had to let it lie repeatedly for entire years, since my innate lethargy, formidably exacerbated by the unstableness of my health, lets me take the pen up only with the greatest difficulty". He goes on to describe what has been done and what is still missing, namely the most important part, "where I teach the application of the principles of *ars conjectandi* to civic, ethical and economical questions". Finally he has solved "a singular problem which has considerable difficulty and even greater usefulness to commend it".[29] His brother, Jacob writes, has known about this main result "for more than twelve years", but when l'Hôpital asked him about it (the 1692 letter in question has been quoted earlier), Johann "hid the truth in his zeal to run my work down". The letter continues with an explicit statement of the problem and Jacob's solution: he has found an "accurate, geometrical proof, which is no mean achievement". His result shows that there is no bound (smaller than certainty) to the degree of probability with which the true ratio of cases can be singled out by increasing the

25 Graunt (1663).

26 de Witt, *Waerdye van Lyf-Renten* (1671).

27 Halley (1693).

28 "Audio a Te doctrinam de aestimandis probabilitatibus (quam ego magni facio) non parum esse excultam. Vellem aliquis varia ludendi genera (in quibus pulchra hujus doctrinae specimina) mathematice tractaret." (*Der Briefwechsel von Jacob Bernoulli*, p. 109).

An English translation of the exchange between Leibniz and Jacob Bernoulli on probability that is paraphrased here and a study in the context of other parts of Leibniz's correspondence can be found in Sylla (1998).

29 "... soluto eum in finem singulari quodam Problemate, quod difficultatis commendationem non parvam, utilitatis longe maximam habet..." (Jacob Bernoulli to Leibniz, Oct. 3rd, 1703: *Der Briefwechsel von Jacob Bernoulli*, p. 116).

number of observations: otherwise the attempt to establish chances empirically would have been doomed, but with the help of his theorem one will actually be able "to investigate this ratio *a posteriori* just as certainly as if it were known *a priori*".

This was the first time Bernoulli had submitted his treasured theorem to a competent judge of its correctness and importance. He must have been very disappointed when Leibniz reacted vaguely and skeptically. "You ask whether a perfect estimation can be obtained this way", he writes; but isn't there too much contingency and variability in events ruled by chance? In most cases, a rough empirical appraisal of the factors involved is anyway sufficient for practical applications.[30] In his next letter from April 1704 Bernoulli tries to insist, giving a numerical instance of the mathematically precise law he has discovered and answering Leibniz's objections; but his appeals for a more considered opinion and for help with locating a copy of de Witt's brochure are unsuccessful. The dialogue closes on a note of resignation.[31]

3. A hard delivery: how did the *Ars Conjectandi* finally come to be published?[32]

Only two months later Jacob Bernoulli's death triggered a wave of interest in his scientific heritage. His former student and assistant lecturer Jacob Hermann supplied the learned journals with material for their obituaries, and both Jacques Saurin in the *Journal des Savants*[33] and Bernard de Fontenelle in the Paris Academy's *Mémoires*[34] gave pride of place to the unfinished *Ars Conjectandi* (strangely, the corresponding paragraph was left out from the obituary in the Leipzig *Acta Eruditorum*,[35] to which it had come through Leibniz). Saurin was the first to voice the hope that "some friendly and skilled hand" would add the necessary complements to the treatise and see it through the press; somewhat naively, he said the learned public expected this "from the goodwill of Mr. Bernoulli the Younger, who has always shown great zeal for the public benefit and the advancement of the sciences and to whom the memory of such a brother must be dear."[36]

However, after all the animosity that had alienated them for many years, Johann had neither the opportunity nor the inclination to act as midwife for his brother's posthumous works. In 1707 he told Varignon in no uncertain terms that Saurin's appeal would come to nothing, both because Jacob's heirs would not let him near the writings left behind and because he had more than enough discoveries of his own to deal with.[37] This is not to say that Johann did not appreciate the importance, in particular, of the *Ars Conjectandi*. In a letter to Leibniz from 1708 he writes that Jacob has "investigated all this more carefully and simply" than in available works on games of chance; and he even takes the very characteristic measure of reserving his own claims on the scientific content: "I have at that time repeatedly contributed my own thoughts, which my brother added to his work". However, he continues, it is perhaps better that the distrust of Jacob's widow and children prevents the papers from falling into his hands, since thus he can avoid any suspicion of intellectual theft.[38]

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30 Cf. Leibniz to Jacob Bernoulli, Nov. 26th, 1703: *Der Briefwechsel von Jacob Bernoulli*, p. 123.

31 Indeed, by June 3rd, 1705, when he wrote to Leibniz for the last time, Jacob Bernoulli had just learnt that his brother was leaving Groningen in order to take over the chair of mathematics at Basel after his death (*Der Briefwechsel von Jacob Bernoulli*, p. 150).

32 Most of this final section relies on Karl Kohli's essay *Zur Publikationsgeschichte der Ars Conjectandi* in Jac. B. *Werke* 3, pp. 391–401.

33 Saurin (1706).

34 Fontenelle (1706).

35 Hermann (1706).

36 "C'est ce qu'ils attendent en particulier des soins de M. Bernoulli le jeune, qui a toûjours fait paroître un grand zele pour l'utilité publique, & pour l'avancement des Sciences, & à qui la mémoire d'un tel frere doit être chere" (Saurin, 1706, p. 89).

37 Joh. I B. to Varignon, Feb. 26th, 1707: cf. *Der Briefwechsel von Johann Bernoulli*, vol. 3, p. 221.

38 Joh. I B. to Leibniz, Sep. 1st, 1708: cf. Gerhardt (1856/1859), vol. III, p. 837.

Jacob's widow, Judith Stupanus, seems to have been a shrewd business woman, but she neither understood nor cared about intellectual property and the fact that its interest depends on priority and thus on its public availability. In 1706 it was planned to send the manuscript heritage along with Jacob's son, an art student, to Varignon in Paris and let him advise the family about publication,[39] but this came to nothing.

The family's caution was not altogether misplaced: at least one of their relatives actually made rather free use of what he had learnt from Jacob's teaching. The last family member we're going to meet in this paper – conventionally called Nicolaus I to distinguish him from two cousins of the same name – was the son of a brother of Jacob and Johann who made his living as a painter. Nicolaus I had studied with Jacob, defending for his master's thesis one part of his uncle's treatise on infinite series,[40] but then went on to Groningen to continue his studies with Johann.

Now he was back in Basel studying law; and on June 14th, 1709, he presented to the local faculty his doctoral dissertation *De usu artis conjectandi in jure*.[41] This did not just rely on the methods developed by Jacob, but except for some rhetorical flourishes and lawyer's lingo, a great part of the arguments and examples were lifted from Jacob's papers almost literally.

Moreover, Johann and Nicolaus were leading an intense correspondence with the French nobleman and amateur mathematician Pierre Rémond de Montmort, who had just published his own book on games of chance[42] and was avidly interested in the *Ars Conjectandi* since he had heard of it. In 1710 he appealed to Johann to help with obtaining the manuscript in trust for publication at his own cost, or at least with getting a look at Part IV for use in a revised edition of his own book.[43]

The second edition of Montmort's *Essay d'analyse sur les jeux de hazard* contains a section of 130 pages reproducing nine letters exchanged with Johann and Nicolaus Bernoulli.[44] One of these letters even contains a variant of the proof for Jacob's Law of Large Numbers that Nicolaus had communicated to Montmort in January 1713. Later in that crucial year Nicolaus spent several months on Montmort's country estate in Champagne discussing probability theory.

Meanwhile, Nicolaus had also appealed to Jacob Hermann at Padua, and together they managed to convince Jacob's son – then also in Italy – that further delay was not in their interest.[45] The decision to have the *Ars Conjectandi* printed was finally taken. However Jacob's family still did not want to let Johann or Nicolaus near the manuscript, so they engaged for the editorial work and the proofreading first a recently promoted doctor of law, then an unemployed minister; both have often been said not to have understood a lot of what they were preparing for publication.[46]

In May 1713 Varignon urgently appealed to Johann and Nicolaus Bernoulli to lend a hand with the book at the last minute.[47] Nicolaus reluctantly complied, contributing an *Errata* list and a preface where he

[39] See Hermann's letter to Leibniz, July 17th, 1706: Gerhardt (1856/1859), vol. IV, p. 301.

[40] Jac. B. Op. CI (1704), *Positionum de Seriebus Infinitis... Pars Quinta* (*Werke* 4, pp. 127–147).

[41] Bernoulli, Nic. I (1709).

[42] Montmort (1708).

[43] Not all the letters Montmort exchanged with Johann I and Nicolaus I Bernoulli have yet been published. Beyond those that Montmort himself printed in the second edition of his book (see note 44), one letter was incorporated in Johann's *Opera*. The Bernoulli–Montmort letters relating to probability, combinatorics and early attempts at a strategic theory of games are analysed in Julian Henny's thesis (1973), which was reprinted in Jac. B. *Werke* 3, pp. 457–507.

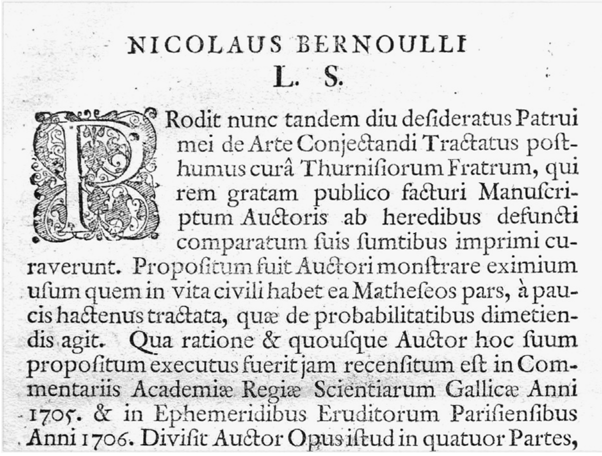
The text of most of Johann I Bernoulli's correspondence – including all the letters to and from Montmort – is available online at the *Basler Edition der Bernoulli-Briefwechsel* (http://www.ub.unibas.ch/bernoulli).

[44] Montmort (1713), pp. 283–414.

[45] Cf. Nic. I B. to Hermann, May 26th, 1711: Ms UB Basel L Ia 21/1, fol. 86 (quoted in Jac. B. *Werke* 3, p. 398).

[46] See, e.g., Joh. I B. to Nic. I B., July 15th, 1712: "... es wird etwas schönes heraus kommen, wenn weder der Editor, noch der Corrector, noch der Trucker nichts von der materie verstehen, und es ohne dem ein unvollständig und gestümpelt werk ist...": Ms UB Basel L Ia 22/1, no. 7 (quoted in Jac. B. *Werke* 3, p. 399).

[47] Varignon to Joh. I B., March 17th, 1713: cf. *Der Briefwechsel von Johann Bernoulli*, vol. 3, p. 534.


NICOLAUS BERNOULLI
L. S.

Rodit nunc tandem diu deſideratus Patrui mei de Arte Conjectandi Tractatus poſthumus curâ Thurniſiorum Fratrum, qui rem gratam publico facturi Manuſcriptum Auctoris ab heredibus defuncti comparatum ſuis ſumtibus imprimi curaverunt. Propoſitum fuit Auctori monſtrare eximium uſum quem in vita civili habet ea Matheſeos pars, à paucis hactenus tractata, quæ de probabilitatibus dimetiendis agit. Qua ratione & quouſque Auctor hoc ſuum propoſitum executus fuerit jam recenſitum eſt in Commentariis Academiæ Regiæ Scientiarum Gallicæ Anni 1705. & in Ephemeridibus Eruditorum Pariſienſibus Anni 1706. Diviſit Auctor Opus iſtud in quatuor Partes,

Figure 8. *Ars Conjectandi*, preface by Nicolaus I Bernoulli (excerpt).

discreetly explains the peculiar circumstances surrounding the late publication of his uncle's masterpiece (cf. Figure 8).[48]

These three pages are the only part of the volume with which he had anything to do; thus it is a massive overstatement if he is still occasionally dubbed the editor of the *Ars Conjectandi*.

Meanwhile, Montmort had tried to speed up the printing of his own book's second edition in order to anticipate the *Ars Conjectandi* on the market, but without success: the permission for publication was only granted in December 1713.[49] On September 9th, Nicolaus had told him that Jacob Bernoulli's book was now out but that "there would be hardly anything new" in it for their circle of experts.[50]

This indeed sums up the consequences of the *Ars Conjectandi*'s belated delivery: the small community of scholars who were able to understand Jacob Bernoulli's innovative ideas and who could possibly have developed them further had by that time already seen much of the technical results in publications by Montmort and de Moivre, where the point of view is, however, again almost exclusively restricted to gambling issues.

For a long time, almost nobody seems to have realised the conceptual richness and the potential for wider perspectives inherent in Bernoulli's stochastics. It would take almost exactly a century until that thread was taken up in Laplace's *Théorie analytique des probabilités*[51] and entwined with the one that Thomas Bayes had spun – also in a posthumous work; but that is another story.

Despite the hardship surrounding its birth, the theory first sketched in the *Ars Conjectandi* has enjoyed a long and vigorous life: in the 21st century, the investigation of all things uncertain still goes by the name "stochastics" that Jacob Bernoulli coined three hundred years ago.

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48 *Ars Conjectandi* (1713), pp. [III–IV] following the title page: "Nicolaus Bernoulli L[ectori] S[alutem]"; "*Errata*" page at the end of the volume, after the reprint of the dissertations on series and the *Lettre à un Amy, sur les Parties du Jeu de Paume*.

49 See Montmort (1713), pp. 415–416.

50 "Il n'y aura gueres rien de nouveau pour vous" (quoted from Montmort, 1713, p. 401).

51 Laplace (1812).

this talk, Edith Dudley Sylla, Victor Pérez Abreu, Ivo Schneider and Stephen Stigler for motivating discussions, the personnel of the Basel University Library and my colleagues Sulamith Gehr and Fritz Nagel for their tireless help with research and presentation.

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[52] In the notes, three members of the Bernoulli family are referred to by the following conventional abbreviations: Jac. B. for Jacob (I) Bernoulli (1654–1705), Joh. I B. for Johann I Bernoulli (1667–1748), Nic. I B. for Nicolaus I Bernoulli (1687–1759).

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